Homogeneous Bands

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Homogeneity

Definition

A countable (first order) structure $\mathcal{M}$ is *homogeneous* if every isomorphism between finitely generated substructures extends to an automorphism of $\mathcal{M}$.

Motivation:

- A structure $\mathcal{M}$ is uniformly locally finite (ULF) if there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that every n-generated substructure has cardinality at most $f(n)$.
- A ULF homogeneous structure is $\aleph_0$-categorical.

Some key classifications

- (Droste, Kuske, Truss (1999)) A non-trivial homogeneous (lower) semilattice is isomorphic to either $(\mathbb{Q}, <)$, the universal semilattice or a homogeneous semilinear order.

- (Schmerl (1979)) Classified posets:
  i) $\mathcal{A}_n$, the antichain of n elements;
  ii) $\mathcal{B}_n$, the union of n incomparable copies of $\mathbb{Q}$;
  iii) $\mathcal{C}_n = \mathcal{A}_n \times \mathbb{Q}$ with partial order

  $$(a, p) < (b, q) \text{ if and only if } p < q \text{ in } \mathbb{Q};$$

  iv) $\mathbb{P}$, the generic poset,

  where $n \in \mathbb{N}^* = \mathbb{N} \cup \{\aleph_0\}$.

- We can recognise $\mathbb{P}$ by the property: if A, B and C are pairwise disjoint finite subsets such that $A < B$, no element of A is above an element of C, and no element of B is below an element of C, then there exists a point z with $A < z < B$ and incomparable with C.

Semigroup basics

- An element e is an **idempotent** if $e^2 = e$. A **band** B is a semigroup in which every element is an idempotent.

- We may define a partial order $\leq$ on B, known as the **natural order**, by
$$e \leq f \Leftrightarrow ef = fe = e.$$

- The Greens relations on a band simplify to:
$$e \,\mathcal{R}\, f \Leftrightarrow ef = f, fe = e;$$
$$e \,\mathcal{L}\, f \Leftrightarrow ef = e, fe = f;$$
$$e \,\mathcal{D}\, f \Leftrightarrow efe = e, fef = f.$$

Motivating question: Given a homogeneous poset P, does there exist a homogeneous band B such that $(B, <)$ is isomorphic to P?

Homogeneous semilattices

- A **semilattice** is a commutative band.
- A **lower semilattice** $(E, <)$ is a poset in which the meet $\wedge$ of any pair of elements exists.
- If Y is a semilattice, then $(Y, <)$ is a lower semilattice. Conversely, given a lower semilattice, we may form a semilattice $(Y, \wedge)$ by defining $a \wedge b$ as the greatest lower bound of $\{a, b\}$.

Lemma (TQG)

Let $(Y, \wedge)$ be a semilattice. Then the following are equivalent:

i) $(Y, \wedge)$ *is a homogeneous semigroup;*

ii) $(Y, <)$ *is a homogeneous lower semilattice.*

Rectangular bands

- A **rectangular band** is a band B satisfying

$$efe = e \text{ for all } e, f \in B.$$

- A rectangular band with a single $\mathcal{R}$-class ($\mathcal{L}$-class) is called a **right (left) zero band**.

Proposition

Let I and J be arbitrary sets. Then $B_{I,J} = (I \times J, \cdot)$ forms a rectangular band, with operation given by

$$(i, j) \cdot (k, \ell) = (i, \ell).$$

Moreover every rectangular band is isomorphic to some $B_{I,J}$. The natural order on $B_{I,J}$ is an anti-chain on $|I| \cdot |J|$ elements, and the Greens relations simplify to:

$$(i, j)\,\mathcal{R}\,(k, \ell) \Leftrightarrow i = k \text{ and } (i, j)\,\mathcal{L}\,(k, \ell) \Leftrightarrow j = \ell.$$

Homogeneous rectangular bands

Proposition

A pair of bands $B_{I,J}$ and $B_{I',J'}$ are isomorphic if and only if $|I| = |I'|$ and $|J| = |J'|$. Moreover if $\phi_I : I \to I'$ and $\phi_J : J \to J'$ are a pair of bijections, then $\phi : B_{I,J} \to B_{I',J'}$ defined by

$$(i,j)\phi = (i\phi_I, j\phi_J)$$

is an isomorphism, and every isomorphism from $B_{I,J}$ to $B_{I',J'}$ can be constructed in this way.

We may thus denote $B_{\kappa,\delta}$ to be the unique (up to isomorphism) rectangular band with κ $\mathcal{R}$-classes and δ $\mathcal{L}$-classes.

Corollary

Rectangular bands are homogeneous. Moreover any homogeneous band B such that $(B,<) \cong \mathcal{A}_n$ is isomorphic to some $B_{i,j}$, where $i \cdot j = n$.

General bands

- While there exists a classification theorem for general bands, it is far too complex for use. Moreover, no general isomorphism theorem exists, so its usefulness in understanding homogeneous bands is minimal. However a weaker form of the theorem will be of use:

Theorem

Let B be an arbitrary band. Then $Y = S/\mathcal{D}$ is a semilattice and B is a semilattice of rectangular bands B_α (which are the $\mathcal{D}$-classes), that is,

$$B = \bigcup_{\alpha \in Y} B_\alpha \text{ and } B_\alpha B_\beta \subseteq B_{\alpha\beta}.$$

- We therefore understand the *global* structure of any band, but not the local structure.

Substructure of homogeneous bands

Lemma (TQG)

If $B = \bigcup_{\alpha \in Y} B_\alpha$ is a homogeneous band, then:

i) *$Aut\ (B)$ is transitive on B, that is if $e, f \in B$ then there exists $\theta \in Aut(B)$ such that $e\theta = f$;*

ii) *Y is homogeneous;*

iii) *$B_\alpha \cong B_\beta$ for all $\alpha, \beta \in Y$.*

- However homogeneity does not pass to all induced substructures of B. For example take B to be the band corresponding to a homogeneous semilinear order. Then the poset $(B, <)$ is not homogeneous.

- Understanding how the rectangular bands interact in a band is thus key to homogeneity.

Poset 2: $\mathcal{B}_n$

- Suppose now that $B = \bigcup_{\alpha \in Y} B_\alpha$ is such that $(B, <) \cong \mathcal{B}_n$. Then B satisfies the following condition: for each e_α and $\beta \leq \alpha$, there exists a unique $e_\beta \in B_\beta$ such that $e_\beta < e_\alpha$. Indeed if $e_\alpha > e_\beta, f_\beta$, then $\{e_\alpha, e_\beta, f_\beta\}$ forms a non-linear, non-antichain, and thus is not embeddable in $\mathcal{B}_n$.

- A **normal band** is a band B satisfying

$$zxyz = zyxz \text{ for all } x, y, z \in B.$$

  This is equivalent to B satisfying the condition above.

- A band B is called a **left/right normal band** if it is normal and each B_α is a left/right-zero band.

Strong semilattices

- Let Y be a semilattice, and $\{B_\alpha : \alpha \in Y\}$ be a collection of disjoint rectangular bands. For each $\alpha \geq \beta$ in Y, let $\phi_{\alpha,\beta} : B_\alpha \to B_\beta$ be a morphism such that:

  i) $(\forall \alpha \in Y)\ \phi_{\alpha,\alpha} = 1_{B_\alpha}$;

  ii) for all $\alpha \geq \beta \geq \gamma$ in Y,

$$\phi_{\alpha,\beta}\phi_{\beta,\gamma} = \phi_{\alpha,\gamma}.$$

  Define multiplication on $B = \bigcup_{\alpha \in Y} B_\alpha$ by the rule that, for each $e \in B_\alpha$, $f \in B_\beta$,

$$ef = (e\phi_{\alpha,\alpha\beta})(f\phi_{\beta,\alpha\beta}).$$

  Then B forms a band, called a **strong semilattice of rectangular bands**, and denoted $[Y, B_\alpha, \phi_{\alpha,\beta}]$.

Proposition

A band is normal if and only if it is isomorphic to a strong semilattice of rectangular bands.

Isomorphism theorem for normal bands

- Not only do normal bands have a structure theorem that allows us to understand the local structure, but vitally there exists an isomorphism theorem:

Theorem

Let $B = [Y, B_\alpha, \phi_{\alpha,\beta}]$ *and* $B' = [Z, B'_\alpha, \psi_{\alpha,\beta}]$ *be normal bands. Let* $\pi : Y \to Z$ *be an isomorphism, and for every* $\alpha \in Y$*, let* $\theta_\alpha : B_\alpha \to B'_{\alpha\pi}$ *be an isomorphism such that for any* $\alpha \geq \beta$ *in* Y*, the diagram*

$$\begin{array}{ccc} B_\alpha & \xrightarrow{\theta_\alpha} & B'_{\alpha\pi} \\ \downarrow{\scriptstyle \phi_{\alpha,\beta}} & & \downarrow{\scriptstyle \psi_{\alpha\pi,\beta\pi}} \\ B_\beta & \xrightarrow{\theta_\beta} & B'_{\beta\pi} \end{array}$$

commutes. Then $\theta = \bigcup_{\alpha \in Y} \theta_\alpha$ *is an isomorphism of* B *into* B'*, denoted* $[\theta_\alpha, \pi]$*. Conversely, every isomorphism of* B *into* B' *can be so obtained for unique* π *and* θ_α*.*

Homogeneous normal bands

- Let $B = [Y, B_\alpha, \phi_{\alpha,\beta}]$ be a normal band with each B_α isomorphic to $B_{n,m}$ for some (fixed) $n, m \in \mathbb{N}^*$.

Lemma (TQG)

If B is homogeneous then each $\phi_{\alpha,\beta}$ is surjective. Moreover, if any $\phi_{\alpha,\beta}$ is an isomorphism, then $B \cong Y \times B_{n,m}$.

Lemma (TQG)

The band $Y \times B_{n,m}$ is homogeneous if and only if Y is homogeneous. Moreover $(Y \times B_{n,m}, \leq)$ is isomorphic to nm incomparable copies of Y.

Corollary

A band B is homogeneous and is such that $(B, <) \cong \mathcal{B}_n$ if and only if $B \cong \mathbb{Q} \times B_{i,j}$, where $i \cdot j = n$.

JEP and AP

- To consider the case where the connecting morphisms are not injective, we turn to a method of model theory; Fraïssé's Theorem.
- Let $\mathcal{K}$ be a class of structures.
- We say that $\mathcal{K}$ has the **joint embedding property** (JEP) if given $B_1, B_2 \in \mathcal{K}$, then there exists $C \in \mathcal{K}$ and embeddings $f_i : B_i \to C$.
- We say that $\mathcal{K}$ has the **amalgamation property** (AP) if given $A, B_1, B_2 \in \mathcal{K}$ (where $A \neq \emptyset$) and embeddings $f_i : A \to B_i$, then there exists $D \in \mathcal{K}$ and embeddings $g_i : B_i \to D$ such that

$$f_1 \circ g_1 = f_2 \circ g_2.$$

Fraïssé's Theorem

Theorem (Fraïssé's theorem)

Let L be a countable signature and let $\mathcal{K}$ be a non-empty finite or countable set of f.g. L-structures which is closed under induced substructures and satisfies JEP and AP.
Then there is an L-structure D, unique up to isomorphism, such that $|D| \leq \aleph_0$, $\mathcal{K}$ is the age of D and D is homogeneous. We call D the **Fraïssé limit** *of $\mathcal{K}$.*

Example 1: The class of all finite rectangular bands, $\mathcal{K}_{RB}$, forms a Fraïssé class, with Fraïssé limit $B_{\aleph_0,\aleph_0}$.

Example 2: Let $\mathcal{K}$ be the class of all finite bands. Since the class of all bands forms a variety, $\mathcal{K}$ is closed under both substructure and (finite) direct product, and thus has JEP. However T. Imaoka showed in 1976 that AP does not hold.

Normal bands

Proposition

The classes $\mathcal{K}_N$, $\mathcal{K}_{RN}$ and $\mathcal{K}_{LN}$ of all finite normal, right normal and left normal bands respectively form Fraïssé classes. Their Fraïssé limits will be denoted B_N, B_{RN} and B_{LN}, respectively.

Lemma (TQG)

Let $B_N = [Y, B_\alpha, \phi_{\alpha,\beta}]$ be the generic normal band. Then

i) *Y is the universal semilattice;*

ii) *$(B, <) \ncong \mathbb{P}$;*

iii) *$B_\alpha \cong B_{\aleph_0,\aleph_0}$ for all $\alpha \in Y$;*

iv) *$\phi_{\alpha,\beta}$ is surjective but not injective for all $\alpha \in Y$;*

Proof.

ii) Let $e_\alpha, f_\alpha \in B_\alpha$. Then $e_\alpha \perp f_\alpha$, and there does not exist $g \in B$ such that $g > e_\alpha, f_\alpha$ since B is normal. Clearly this cannot hold in $\mathbb{P}$. □

Homogeneous bands over $\mathbb{Q}$

- Let ρ be an equivalence relation on a band B. We say that B satisfies ρ-**covering** if for any $e, f, g \in B$,

$$e > f \text{ and } f \,\rho\, g \Rightarrow e > g.$$

- For example if $B = \bigcup_{\alpha \in \mathbb{Q}} B_\alpha$ satisfies $\mathcal{D}$-covering then for any $\alpha > \beta$, $e \in B_\alpha$ and $f \in B_\beta$ we have $e > f$.

Lemma (TQG)

Let $B = \bigcup_{\alpha \in \mathbb{Q}} B_\alpha$ be a band satisfying ρ-covering, where $\rho = \mathcal{D}, \mathcal{R}$ or $\mathcal{L}$. Then B is homogeneous if and only if $B_\alpha \cong B_\beta$ for all $\alpha, \beta \in \mathbb{Q}$. Moreover, if $\rho = \mathcal{D}$ and B is homogeneous then $(B, <)$ is isomorphic to the homogeneous poset $(\mathcal{A}_n \times \mathbb{Q}, <)$, where $n = |B_\alpha|$.

Homogeneous bands over $\mathbb{Q}$

Lemma

Let $B = \bigcup_{\alpha \in \mathbb{Q}} B_\alpha$ and $C = \bigcup_{\alpha \in \mathbb{Q}} C_\alpha$ be a pair of homogeneous bands satisfying ρ-covering, where $\rho = \mathcal{D}, \mathcal{R}$ or $\mathcal{L}$. Then $B \cong C$ if and only if $B_\alpha \cong C_\alpha$.

- We may thus denote $D_{n,m}, R_{n,m}$ and $L_{n,m}$ as the unique (up to isomorphism) homogeneous band with $\mathcal{D}, \mathcal{R}$ and $\mathcal{L}$-covering respectively and $\mathcal{D}$-classes isomorphic to $B_{n,m}$.

Corollary (TQG)

A band $B = \bigcup_{\alpha \in \mathbb{Q}} B_\alpha$ is homogeneous if and only if isomorphic to either

i) $B_{n,m} \times \mathbb{Q}$;

ii) $D_{n,m}, R_{n,m}$ or $L_{n,m}$,

for some $n, m \in \mathbb{N}^*$.

The other cases

Let $B = \bigcup_{\alpha \in Y} B_\alpha$ be a homogeneous band.

Proposition (TQG)

If Y is a non-linear semilinear order then B is normal.

Lemma (TQG)

If Y is the universal semilattice and B is not normal then for any $e, f \in B_\alpha$ we have

$$eBe \setminus \{e\} = \{g \in B : g < e\} = \{g \in B : g < f\} = fBf \setminus \{f\}.$$

Moreover $(B, <) \ncong \mathbb{P}$.

Open problem: Are homogeneous bands over the universal semilattice necessarily normal?

Summary

Proposition

The following bands are homogeneous:

i) *Generic type:* B_N, B_{RN} *and* B_{LN}*;*

ii) $Y \times B_{n,m}$ *where* Y *is a homogeneous semilattice;*

iii) $D_{n,m}, R_{n,m}$ *and* $L_{n,m}$*,*

for any $n, m \in \mathbb{N}^*$*. Moreover if* P *is a homogeneous poset then there exists a homogeneous band* B *such that* $(B, <) \cong P$ *if and only if* $P \ncong \mathbb{P}$*.*

Note: Given a homogeneous poset $P \ncong \mathbb{P}$, the existence of a homogeneous band B such that $(B, <) \cong P$ is not unique in general. In fact B is unique up to isomorphism if and only if P is trivial or $(\mathbb{Q}, <)$.